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A-infinity structure and superpotentials

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Abstract: A_∞ algebras and categories are known to be the algebraic structures behind open string field theories. In this note we comment on the relevance of the homology construction of A_∞ categories to superpotentials.

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Branes in Calabi-Yau manifolds are an important arena for uncovering non-perturbative features of string theory, with interesting mathematical phenomena as by-products or important ingredients. Such a sentence can be partially justified noting the remarkable fact that mathematicians, trying to uncover the core of the mirror symmetry phenomena, came to formulate a conjecture in terms of derived categories first [1], and later of A_∞ ones [2]. The physical reader of such conjectures comes immediately to think that the latter have to do in some way with string field theory, in which A_∞ was recognized to emerge some time ago [3]. This note is a first try to make this connection more precise, applying it to the problem of superpotentials in Calabi-Yau compactifications [4, 5].

A_∞ and mirror symmetry. Everything starts from the study of twisted topological models and their boundary conditions [7]. Remarkably, A and B branes make an *ante litteram* appearance there. This is one of the main reasons leading to a conjecture [1] meant to explain the "mathematical mysteries" of mirror symmetry. This *homological mirror symmetry* conjecture states the equivalence of the derived category of coherent sheaves on a Calabi-Yau Y and of the derived category of the Fukaya category of the mirror $\tilde{Y}$. This relates respectively B branes on Y with A branes on $\tilde{Y}$.

But, Fukaya category is an A_∞ category. Since A_∞ structure is important in string field theory, to a physicist this should already suggest the existence of a refined conjecture making reference directly to the Fukaya category itself, without having to associate a derived category to it. Remarkably, even without this string field theory suggestion, mathematicians came to the same conclusion, formulating (and checking in one example) exactly this refined conjecture [2], building on previous general work on how to define an A_∞ structure from differential graded algebras [8].

What a physicist can do now is to try to use this *explicit* A_∞ structure to do computations in string theory. What we know is that a string field theory action S obeying Batalin-Vilkovisky master equation $\{S, S\} = 0$ can be written from an A_∞ structure and a bilinear form. Reviewing this requires a crash course on A_∞ structures.

A_∞ and string field theory. Let us start from A_∞ algebras [6]. These are generalizations of differential associative algebras, in which the product is not required to be associative. A clever way to introduce them is the so-called bar construction [6, 9], whose relevance in string theory would be interesting to elucidate. We will instead be more down to earth and define A_∞ algebra as a $\mathbb{Z}$-graded vector space $\mathcal{A}$, endowed with linear maps

$$m_k : \underbrace{\mathcal{A} \otimes \cdots \otimes \mathcal{A}}_{k \text{ times}} \to \mathcal{A}$$

of grade $2-k$, satisfying an infinite set of conditions, among which we display only the first ones:

$$
\begin{aligned}
& m_1^2 = 0\,; \\
& m_1(m_2(\phi_1,\phi_2)) = m_2(m_1(\phi_1),\phi_2) + (-)^{|\phi_1|} m_2(\phi_1, m_1(\phi_2))\,; \\
& m_2(\phi_1, m_2(\phi_2,\phi_3)) - m_2(m_2(\phi_1,\phi_2),\phi_3) = \\
& \quad = m_1(m_3(\phi_1,\phi_2,\phi_3)) + + m_3(m_1(\phi_1),\phi_2,\phi_3) + (-)^{|\phi_1|} m_3(\phi_1, m_1(\phi_2),\phi_3) + \\
& \qquad + (-)^{|\phi_1|+|\phi_2|} m_3(\phi_1,\phi_2,m_1(\phi_3)) \ ,
\end{aligned} \tag{1}
$$

with $\phi_i \in \mathcal{A}$, and $|\phi_i|$ their grades. Second equation shows m_1 is a differential; m_2 can be thought of as a multiplication which is not associative but almost so, as measured by the presence of the terms with m_3 in (1).

If one has such a structure on the Hilbert space of a string theory (and a symplectic bilinear form $\langle\,,\rangle$), one can define the promised string field theory action as

$$
S = \frac{1}{2}\langle \Phi, Q\Phi\rangle + \frac{1}{3}\langle \Phi, b_2(\Phi,\Phi)\rangle + \frac{1}{4}\langle \Phi, b_3(\Phi,\Phi,\Phi)\rangle + \cdots \,, \tag{2}
$$

where actually the b_k are not the m_k but close relatives [9, 3], and Φ is the string field. One can recognize in this formula the differential m_1 as being the BRST operator Q, and m_2 as being the string field theory product.

A similar work, generalizing associative structure to A_∞ one, can be done with linear categories. The idea of linear categories is just to relax the usual conditions on product structures in algebra, that require the product to be there *for any* couple of elements. The way to do this is to introduce first a class of useful labels, called *objects*. Then, one introduces for any pair of objects a, b, a vector space of *morphisms* $\mathrm{Hom}(a,b)$. These are the generalization of the elements of the algebra: there is a product $\circ$ such that a $\phi \in \mathrm{Hom}(a,b)$ and a $\phi' \in \mathrm{Hom}(a',b')$ can be multiplied only if $b = a'$: one can imagine the morphisms as arrows, and say that the product is defined only when the head of the first arrow coincides with the tail of the second one. With this important proviso, which is the whole difference between a category and an algebra, all the rest of the definition remains the same: namely, in all the $\mathrm{Hom}(a,b)$ it is required to have an unity element; there should be associativity $(\phi\circ\phi')\circ\phi'' = \phi\circ(\phi'\circ\phi'')$ whenever the products are defined; and $\circ$ should be bilinear in the two entries (distributive law).

Now, as for algebras, one can define an A_∞ category by replacing associative multiplication with maps

$$
m_k : \mathrm{Hom}(B_1,B_2)\otimes \mathrm{Hom}(B_2,B_3)\otimes\cdots\otimes \mathrm{Hom}(B_{k-1},B_k) \to \mathrm{Hom}(B_1,B_k)
$$

obeying conditions (1) whenever the multiplications are defined. This is relevant for physics again thanks to the applications to string field theory [10]. Indeed, branes form categories: This means that the objects of the category are branes, and the morphisms between them are the states in the Hilbert space of open strings connecting them. In general this category will be an A_∞ one; then the analogue of (2) can be defined in this case [10], using the more refined A_∞ category structure instead of the A_∞ algebra structure. It is more refined in the sense that, given a category, one can always do a trick and consider the algebra obtained "collapsing all objects to one"; namely, defining a product also for arrows whose head and tail do not coincide, as a zero product. One can do so, but the category picture displays more structure.

Homology and superpotentials. We have said branes form a category. If one considers off-shell states, the category will sometimes be associative (as in the case of topological models), but in general will be an A_∞ one. For any A_∞ algebra $\mathcal{A}$, we can define its *homology* $H^*\mathcal{A}$, whose objects are the same as those of $\mathcal{A}$ and whose morphisms are the cohomology of $\mathcal{A}$ with respect to the differential m_1. This is again an A_∞ algebra [9]; if $\mathcal{A}$ is a differential graded algebra this can be explicitly shown [11], as we are going to review in a moment. Again, all these things can be generalized to categories.

Physically, $H^*\mathcal{A}$ is the Hilbert space of physical states, and this is where things start being interesting. Let us consider a definite case: B branes in a Calabi-Yau threefold Y. We know that the problems of classifying such states at a general point in moduli space decouples in a holomorphic problem, coming from an F-flatness in 4d terminology, and a more difficult stability problem, coming from the D-term; they depend on holomorphic and Kähler moduli respectively [4]. To talk about the holomorphic side without having to worry about stability, a standard trick is to consider topologically B-twisted model. In this case, string field theory action has been shown [7] to be associative, and more precisely to have the form of holomorphic CS action

$$S = \int \Omega \wedge \left(\frac{1}{2} A \wedge \bar{\partial} A + \frac{1}{3} A \wedge A \wedge A \right) \tag{3}$$

for space filling branes, with A being $(0,1)$ connection. In the non-space-filling cases, one takes dimensional reduction, as for instance in [5].

Now, the crucial idea is this. The string field theory (2) action is determined by the algebraic structure (the m_k). This action gives in principle a means to compute all of the correlation functions. What is the meaning of the algebraic structure on the category of physical states, then? They will be correlation functions of the physical states. Since this is the same as computing superpotentials (see for example [4] for details), we can then interpret these as giving term by term the complete superpotential of the theory.

Let us come back to our example, and compute this induced A_∞ structure on the physical states. For simplicity we start taking two space-filling branes B_0 and B_1. The power of this method is however that it can be extended to coherent sheaves and more general elements of the derived category; in perspective, this can be useful away from the large volume limit, where one expects that "strange" elements of the derived category will become stable. The starting Hilbert space is simply [7] $\Omega^{(0,p)}(B_1^* \otimes B_2)$ with its structure of differential graded algebra, the differential being $\bar\partial_{B_1^*\otimes B_2} \equiv \bar\partial$ and the product being the $\wedge$ product between forms. The physical states are then the cohomology of the $\bar\partial$ operator. Note that the holomorphic CS action is simply the application of (2) to this algebra, and this tells us that in this case the symplectic form on string field algebra is $\int \Omega(\cdot \wedge \cdot)$.

The A_∞ structure on this algebra is now given as follows: Consider the operator $\mathcal{H} \equiv 1 - \Delta_{\bar\partial} G - G\Delta_{\bar\partial}$. This is the projector on harmonic forms [12, 8, 2]. We recall that harmonic forms are a slice of the cohomology, in the sense that there is exactly one harmonic form in any cohomology class. This is familiar: they satisfy $\bar\partial\omega = 0 = \bar\partial^\dagger\omega$; first condition is their being closed, second one "fixes the gauge" of the invariance $\omega \cong \omega + \bar\partial\alpha$. The vector space on which we will impose the A_∞ structure is thus the cohomology of the complex, or equivalently the subcomplex of harmonic forms, $\mathrm{Ext}^p(B_1, B_2)$. Now, the products. The differential m_1 is taken to be zero (it is $\bar\partial$ restricted to harmonic forms). The product m_2 is simply the harmonic part of the wedge product:

$$m_2(A_1, A_2) = \mathcal{H}(A_1 \wedge A_2)\,.$$

Higher products are less obvious, but are the interesting part. They can be written iteratively in terms of the operator $p \equiv \bar\partial^\dagger G$ (where G is Green function for the the laplacian $\Delta_{\bar\partial}$ as [8]

$$\begin{aligned} m_k(A_1, \ldots, A_k) \equiv \mathcal{H}\Big\{ & (-)^{k-1}[p\, m_{k-1}(A_1, \ldots, A_{k-1})] \wedge A_k - (-)^{k\, a_1} A_1 \wedge \\ & \wedge [p\, m_{k-1}(A_1, \ldots, A_{k-1})] - \sum_{\substack{i+j=k+1 \\ i,j\geq 2}} (-1)^{i+(j-1)(a_1+\ldots+a_i)} \times \\ & \times [p\, m_i(A_1, \ldots, A_i)] \wedge [p\, m_j(A_{j+1}, \ldots, A_k)] \Big\}\,, \end{aligned} \tag{4}$$

where a_i is the degree of A_i. Having such products in our hands, the superpotentials will be now, exactly in analogy with the string field action (3), and based on what we have said before,

$$W = \int \Omega \wedge \mathrm{tr} \left\{ A \wedge (\frac{1}{3} A \wedge A + \frac{1}{4} m_3(A, A, A) + \frac{1}{5} m_4(A, A, A) + \ldots) \right\}, \tag{5}$$

where A are now *harmonic* $(0,1)$ forms. Actually, more precisely, these A have to be understood as elements of $\mathrm{Ext}^1(B_1, B_2)$, though we will keep the notation for forms. In particular, for lower-dimensional branes some of these A have to be understood actually as transverse scalars X, as we will see more precisely later. Note also that trace in (5) is simply circular matching of gauge indices of the various "bifundamentals" corresponding to strings stretched between B_1 and B_2, between B_2 and B_3,..., between B_k and B_1.

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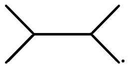
Let us see how do really these higher products look like by working out m_3. The corresponding piece of the superpotential is simply of the form (in a little schematic way again, for the time being)

$$\int \Omega \wedge \mathrm{tr}\left\{A \wedge A \wedge \bar{\partial}^{\dagger}(G\, A \wedge A)\right\}; \tag{6}$$

we have written the operator p explicitly to underline the structure of this piece. G can be understood roughly as Δ^{-1}, and so the operator $\bar{\partial}^{\dagger} G$ is formally a kind of $(\bar{\partial})^{-1}$, that is, the propagator corresponding to the first-order kinetic term $\bar{\partial}$ (or, if one prefers, in string field theory terms this is b_0/L_0). The term (6) can thus be thought of as the Feynman diagram .

One can see a similar Feynman interpretation for all the terms in this superpotential; this is more or less the reinterpretation of the A_∞ construction [8] in terms of trees [13], and is the better justification for our claims: in a sense the A_∞ structure has the role of resumming all the graphs (something similar happens in [14]). It also suggests a more general relevance of A_∞ structure in effective field theories.

In these expressions, again these A should be thought of as being a symbol for several things. We will try now to be more precise. First of all, as we mentioned, for lower-dimensional branes these A should be understood as A's or transverse scalars X's. Both of these are elements of $Ext^1(B, B)$. Mathematically, one can see this in the following way. Consider for simplicity a bundle on a divisor D which is restriction of a bundle E on the ambient manifold Y. Then from the usual exact sequence

$$0 \to E \otimes \mathcal{O}(-D) \to E \to E_{|_D} \to 0$$

and considering the exact sequence of $\mathrm{Ext}^i(\,\cdot\,, E_{|_D})$ we get

$$\cdots \to H^0(D, End(E) \otimes N_{D,Y}) \to \mathrm{Ext}^1(E_{|_D}, E_{|_D}) \to H^1(D, End(E)) \to \cdots;$$

in this sense, Ext^1 is made of these two pieces, transverse scalars and connections, as one already knows from reduction common sense. The fact that we consider here Ext^1 is justified by the fact that these are the internal parts of the chiral multiplets, whose superpotential we are indeed computing.

Apart from this, we should note that in easy situations one can get from this somewhat trivial results; the various terms in each m_k may cancel (we have schematically displayed only one in (6)). In more general situations, however, these will give probably interesting results about obstructions of moduli spaces of curves. Just to have an idea, a typical term arising by reduction on e.g. a curve will be

$$\int \Omega_{ijz} X^i A_{\bar{z}}\, \bar{\partial}^{\dagger}(G X^j A) dz \wedge d\bar{z}\,,$$

where this time we have denoted by A really connections, being instead transverse scalars explicitly denoted by X^i. Let us also note again that this formalism can be applied more generally to $A \in \mathrm{Ext}^1(B_i, B_j)$ with B_i general objects in the derived category [2].

An important direction of development is towards inclusion of closed string theory. The closed string field theory for the B model, in analogy for the open string sector, is known [15] as Kodaira-Spencer theory, because it reproduces the homonymous deformation theory. Coupling between the two encodes deformations of open string field theory [16]. The idea should be to generalize the homology construction to the BV algebra of open-closed string theory; namely, to find another such structure on the homology of the open-closed algebraic structure, generalizing the Kadeishvili theorem we have cited here for A_∞ algebras (or categories). This will probably have again a Feynman diagram interpretation, and would allow us to interpret the result as open-closed superpotentials. Such terms are very interesting because describe the behaviour of embedded families of subvarieties varying complex structure moduli [4].

Though we have dealt so far with B branes, analogous reasonings can be done with three-dimensional lagrangian submanifolds; in that case string field theory action is this time usual Chern-Simons theory, with instanton corrections. $\bar{\partial}$ is now replaced by d, and physical states are now in $H^i(M, End(E))$, where M is the lagrangian submanifold and E is the flat bundle over it. We can repeat now the discussion above; finally we should find contact with Floer homology and A_∞ structure of Fukaya category.

A last point is the following. An important property of A_∞ algebras is their connection with extended moduli spaces: given an associative algebra A, if its second Hochschild $HH^2(A, A)$ parameterizes associative deformations, all of them, $HH^*(A, A)$, parameterize A_∞ deformations. Thus, extended moduli spaces, suggested in mathematics again from mirror symmetry considerations [1], should indeed be relevant for string theory: they should parameterize deformations of the string theory action. For instance, one can deform the very string field theory algebra of the topological models from the differential graded category structure $\mathrm{Ext}^*(B_i, B_j)$ to an A_∞ structure, yielding a new action (through the general formula (2) adapted to brane categories in the spirit of [10]) which is still interpretable as a string theory.

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Note added. While this note was in preparation, a paper appeared [17] in which the same results are derived.

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